Eigenvalues as Building Bricks

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December 2019

Introduction. Let $\mathbb{A}$ be an n-dimensional hermitian matrix. Such objects are fully characterized/distinguished from one another by their spectra $\{\lambda_i\}$ and orthonormal eigenvectors $\{\boldsymbol{e}_i\} : i = 1, 2, \ldots, n$. Spectral decomposition[1]

$$\mathbb{A} = \sum_{i=1}^{n} \lambda_i \mathbb{P}_i \quad : \quad \mathbb{P}_i = \boldsymbol{e}_i \boldsymbol{e}_i^{\mathsf{T}}$$

suggests that the elements $\mathbb{P}_i$ of the complete commuting set $\{\mathbb{P}_i\}$ of hermitian projection matrices are more fundamental than the vectors onto which they project: $\mathbb{P}_i \boldsymbol{e}_i = \boldsymbol{e}_i$. This foundational linear algebra finds important application to the measurement theory of n-state quantum systems. In that context, $\mathbb{A}$ provides representation of a measurement device, the λ_i are the possible post-measurement readings of the device, and—given that the system was in state $\boldsymbol{\psi}$—$\boldsymbol{\psi}^{\mathsf{T}}\mathbb{P}_i\boldsymbol{\psi}$ is the probability that the device reads λ_i.

Peter Denton, Stephen Parke & Xining Zhang (DPZ) are neutrino physicists (working at Fermilab, Chicago & Brookhaven, respectively), with vested interest therefore in a 3-state quantum system. While preparing a recent publication[2] they hit upon and made essential use of a pretty aspect of the linear algebra sketched above that appears to have escaped prior notice—a result that engaged the excited interest of Terrence Tao (UCLA).[3] In Natalie Wolchover's article "Neutrnos lead to unexpected discovery in basic math" (Quanta Magazine, 13 November 2019)—from which I learned of this

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[1] We assume the spectrum to be non-degenerate, the eigenvectors to be column vectors, and use $^{\mathsf{T}}$ to denote conjugate transposition.

[2] "Eigenvalues: the Rosetta Stone for neutrino oscillations in matter," arXiv: 1907.02534v1 [hep-ph] 4 Jul 2019.

[3] MacArthur Fellow and Fields Medalist 2006, prolifically active in a great many fields, especially the theory of random matrices.

development—it is reported that within two hours Tao devised three proofs of the DPZ result. Two of those are presented in the $2\frac{1}{2}$-page paper[4] that the physicists + Tao (DPTZ) posted to the web, from which I work.

In the following discussion I follow the pattern of DPTZ's second argument to produce a generalization of their result.[5] As a matter only of typographic convenience, I will assume $\mathbb{A}$ to be real symmetric (= specialized hermitian), and will set $n = 3$ (manageably small, yet not too small); both restrictions are easily relaxed. It will emerge that the DPTZ result is (as are its generalizations) no more than a corollary Cramer's Rule.

Classic preliminaries. Given

$$\mathbb{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

the a_{ab}-submatrix $\mathbb{A}_{ab}$ is the 2×2 matrix produced by striking the a^{th} row and the b^{th} column, of which there are, in the general case n^2, and in the present instance 9. From

$$\mathbb{A}_{12} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}, \quad \mathbb{A}_{21} = \begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix}$$

we see that if $\mathbb{A} = \mathbb{A}^{\intercal}$ then $\mathbb{A}_{ab} = \mathbb{A}_{ba}{}^{\intercal}$. The *cofactor* of a_{ab} is defined/denoted

$$c_{ab} \equiv (-)^{a+b} \det \mathbb{A}_{ab}$$

and the *comatrix* of $\mathbb{A}$ is the matrix of cofactors:

$$\begin{aligned} \mathbb{C} &= \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \\ &= \begin{pmatrix} +\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} & -\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} & +\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \\ -\det\begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix} & +\det\begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} & -\det\begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix} \\ +\det\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix} & -\det\begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix} & +\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{pmatrix} \end{aligned}$$

From assumed symmetry/hermiticity of $\mathbb{A}$ follows (by $c_{ab} = \bar{c}_{ba}$) the symmetry/hermiticity also of $\mathbb{C}$.

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[4] "Eigenvectors from eigenvalues," arXiv:1906.03795v1 [math.RA] 10 Aug 2019. It will emerge that the title is in one important respect misleading.

[5] Their first argument proceeds from an assumption that, while it may be physically motivated, is mathematically extraneous, and requires development of a "Cauchy-Binet type" lemma.

From Cramer's Rule[6] follows this classic construction of $\mathbb{A}^{-1}$:

$$\mathbb{A}^{-1} = \frac{1}{\det \mathbb{A}} \operatorname{adj} \mathbb{A} \tag{1}$$

where $\operatorname{adj}\mathbb{A}$, the *adjugate* of $\mathbb{A}$, is defined[7]

$$\operatorname{adj}\mathbb{A} = \text{transpose of } \mathbb{C}$$

We will find it convenient in place of (1) to write

$$\operatorname{adj}\mathbb{A} = (\det \mathbb{A})\, \mathbb{A}^{-1} \tag{2}$$

The DPTZ construction. For λ not an eigenvalue of $\mathbb{A}$ we as an instance of (2) have

$$\operatorname{adj}(\lambda\mathbb{I} - \mathbb{A}) = \det(\lambda\mathbb{I} - \mathbb{A}) \cdot (\lambda\mathbb{I} - \mathbb{A})^{-1}$$

From $\det(\lambda\mathbb{I}-\mathbb{A}) = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$ and the fact—for all $f(\bullet)$—that if $\mathbb{A}\boldsymbol{e} = \lambda\boldsymbol{e}$ then $f(\mathbb{A})\boldsymbol{e} = f(\lambda)\boldsymbol{e}$, we have

$$\operatorname{adj}(\lambda\mathbb{I} - \mathbb{A})\boldsymbol{e}_j = \frac{(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)}{(\lambda-\lambda_j)}\,\boldsymbol{e}_j$$

So the orthonormal eigenvectors of $\mathbb{A}$ are eigenvectors also of $\operatorname{adj}(\lambda\mathbb{I} - \mathbb{A})$, with eigenvalues

$$\Lambda_j(\lambda) = \prod_{k=1,k\neq j}^{n} (\lambda - \lambda_k) \tag{3}$$

By spectral decomposition we therefore have

$$\operatorname{adj}(\lambda\mathbb{I} - \mathbb{A}) = \sum_{j=1}^{n} \Lambda_j(\lambda)\mathbb{P}_j \tag{4}$$

with
$$\mathbb{P}_j = \begin{pmatrix} e_{j1}\bar{e}_{j1} & e_{j1}\bar{e}_{j2} & \cdots & e_{j1}\bar{e}_{jn} \\ e_{j2}\bar{e}_{j1} & e_{j2}\bar{e}_{j2} & \cdots & e_{j2}\bar{e}_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ e_{jn}\bar{e}_{j1} & e_{jn}\bar{e}_{j2} & \cdots & e_{jn}\bar{e}_{jn} \end{pmatrix}$$

The components of the eigenvectors $\boldsymbol{e}_j$ are real/complex according as $\mathbb{A}$ is real symmetric or hermitian, so the projection matrices $\mathbb{P}_j$ are real symmetric or hermitian according as $\operatorname{adj}(\lambda\mathbb{I} - \mathbb{A})$ is.

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[6] That is, from the determinental solution (when it exists) of the linear system $\mathbb{A}\boldsymbol{x} = \boldsymbol{y}$ developed by the Genevoise mathematician Gabrial Cramer (1704–1752) in 1750.

[7] For real matrices the distinctoin between "transposition" and "conjugated transposition" disappears, and we have $\operatorname{adj}\mathbb{A} = \mathbb{C}^{\mathsf{T}}$.

Now set $\lambda = \lambda_i$. Get

$$\Lambda_j(\lambda_i) = \begin{cases} \displaystyle\prod_{k=1,k\neq j}^{n} (\lambda_i - \lambda_k) & : \quad j = i \\ 0 & : \quad j \neq i \end{cases}$$

so the sum in (4) collapses to a single term, giving

$$\text{adj}\,(\lambda_i \mathbb{I} - \mathbb{A}) = \Lambda_i(\lambda_i)\mathbb{P}_i = \prod_{k=1,k\neq i}^{n} (\lambda_i - \lambda_k)\,\mathbb{P}_i$$

which element-wise reads

$$e_{ia}\bar{e}_{ib} = \text{adj}\,(\lambda_i \mathbb{I} - \mathbb{A})_{ab} \Big/ \prod_{k=1,k\neq i}^{n} (\lambda_i - \lambda_k) \tag{5}$$

Look now to the numerator of the expression on the right side of (5). We have

$$\mathbb{C}(\lambda) = \begin{pmatrix} c_{11}(\lambda) & c_{12}(\lambda) & c_{13}(\lambda) \\ c_{21}(\lambda) & c_{22}(\lambda) & c_{23}(\lambda) \\ c_{31}(\lambda) & c_{32}(\lambda) & c_{33}(\lambda) \end{pmatrix}$$

with

$$\left.\begin{aligned}
c_{11}(\lambda) &= +\det\begin{pmatrix} \lambda - a_{22} & -a_{23} \\ -a_{32} & \lambda - a_{33} \end{pmatrix} \\
c_{12}(\lambda) &= -\det\begin{pmatrix} -a_{21} & -a_{23} \\ -a_{31} & \lambda - a_{33} \end{pmatrix} \\
c_{13}(\lambda) &= +\det\begin{pmatrix} -a_{21} & \lambda - a_{22} \\ -a_{31} & -a_{32} \end{pmatrix} \\
c_{21}(\lambda) &= -\det\begin{pmatrix} -a_{12} & -a_{13} \\ -a_{32} & \lambda - a_{33} \end{pmatrix} \\
c_{22}(\lambda) &= +\det\begin{pmatrix} \lambda - a_{11} & -a_{13} \\ -a_{31} & \lambda - a_{33} \end{pmatrix} \\
c_{23}(\lambda) &= -\det\begin{pmatrix} \lambda - a_{11} & -a_{12} \\ -a_{31} & -a_{32} \end{pmatrix} \\
c_{31}(\lambda) &= +\det\begin{pmatrix} -a_{12} & -a_{13} \\ \lambda - a_{22} & -a_{23} \end{pmatrix} \\
c_{32}(\lambda) &= -\det\begin{pmatrix} \lambda - a_{11} & -a_{13} \\ -a_{21} & -a_{23} \end{pmatrix} \\
c_{33}(\lambda) &= +\det\begin{pmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{pmatrix}
\end{aligned}\right\} \tag{6}$$

From $a_{ab} = \bar{a}_{ba}$ it is seen to follow that the submatrices $\mathbb{M}_{ab}$ that appear in (6) stand in the relation $\mathbb{M}_{ab} = {\mathbb{M}_{ba}}^{\intercal}$, which gives back $c_{ab}(\lambda) = \bar{c}_{ba}(\lambda)$. We note also that in this 3-dimensional example the elements that stand on the diagonal of $\mathbb{C}(\lambda)$ are quadratic in λ, while the off-diagonal elements are linear in λ. In the n-dimensional case the diagonal elements of $\mathbb{C}(\lambda)$ are polynomials of order λ^n, the off-diagonal elements of order λ^{n-1}.

We are in position now to construct detailed descriptions of the elements of

$$\text{adj}\,(\lambda\mathbb{I}-\mathbb{A}) = \begin{pmatrix} c_{11}(\lambda) & c_{21}(\lambda) & c_{31}(\lambda) \\ c_{12}(\lambda) & c_{22}(\lambda) & c_{32}(\lambda) \\ c_{13}(\lambda) & c_{32}(\lambda) & c_{33}(\lambda) \end{pmatrix} = \text{transposed } \mathbb{C}(\lambda)$$

and therefore of $\text{adj}\,(\lambda_i\mathbb{I}-\mathbb{A})$, and to feed those into (5). Let

$$\omega_{ab,s}(\lambda_i) \quad : \quad s = 1, 2, \ldots, n-1$$

denote the eigenvalues of the submatrices $\mathbb{M}_{ab}(\lambda_i)$. Then (5) can be written

$$e_{ia}\bar{e}_{ib} = \prod_{s=1}^{n-1}\omega_{ba,s}(\lambda_i)\Big/\prod_{k=1,k\neq i}^{n}(\lambda_i-\lambda_k) \tag{7}$$

which provides an eigenvalue-wise description of the eigenvector component products that appear (for example) in the development of $\boldsymbol{e}_i^{\mathsf{T}}\mathbb{B}\,\boldsymbol{e}_i$ ($\mathbb{B}$ any $n\times n$ matrix).

DPTZ have special interest in the *squared norms* $|e_{ia}|^2 = e_{ia}\bar{e}_{ia}$ of the components of $\boldsymbol{e}_i$ which, because they live on the diagonal of $\mathbb{P}_i$, can be developed in finer detail. If $\alpha_{a,s}$ denote the eigenvalues of the submatrices $\mathbb{A}_{aa}$ then

$$\omega_{aa,s}(\lambda_i) = \lambda_i - \alpha_{a,s}$$

and (7) becomes

$$\boxed{|e_{ia}|^2 = \prod_{s=1}^{n-1}(\lambda_i-\alpha_{a,s})\Big/\prod_{k=1,k\neq i}^{n}(\lambda_i-\lambda_k)} \tag{8}$$

which is DPTZ's main result, to which they give no name, but might be called the **spectral construction theorem**. Though of novel appearance, it has been seen to derive from familiar stuff: Cramer's Rule and the spectral decomposition of hermitian matrices.

Limitations. DPTZ do not mention—nor will I attempt to address—the modifications to which (8) must be subjected in cases where the spectrum of $\mathbb{A}$ is degenerate.

Let a real symmetric (or—a bit less conveniently—a hermitian) matrix $\mathbb{A}$ be given. It is then easy, with a resource like *Mathematica*, to *verify* (8), but one wonders What is the point? It is easy enough in such cases to construct the eigenvalues and normalized eigenvectors *directly*. If one does use (8) to obtain the squared norms of the components of $\boldsymbol{e}_i$, those in themselves are sufficient to determine only that

$$\boldsymbol{e}_i = \begin{pmatrix} e^{i\varphi_1}\sqrt{|e_{i1}|^2} \\ e^{i\varphi_2}\sqrt{|e_{i2}|^2} \\ \vdots \\ e^{i\varphi_n}\sqrt{|e_{in}|^2} \end{pmatrix}$$

and we lack means to determine the $n-1$ relative phases; in real symmetric cases we lack means (except by trial and error) to select correctly from the $\pm$ sign alternatives. In this respect the title of the DPTZ paper[4] is a bit misleading.

Orthonormality & completeness. The Spectral Construction Theorem appears not to supply information of the type $e_{ia}\bar{e}_{ja}$ required to establish orthogonality of the eigenvectors. But it should on that basis be possible—as a consistency check—to confirm normality. I look by way of illustration to the normality of $\boldsymbol{e}_1$ in cases where the 3-dimensional real symmetric matrix $\mathbb{A}$ has a non-degenerate spectrum.

Our objective is to establish that $\sum_{a=1}^{3}|e_{1a}|^2 = 1$. By (8) we in this instance have

$$\begin{aligned} |e_{11}|^2 &= \frac{(\lambda_1-\alpha_{1,1})(\lambda_1-\alpha_{1,2})}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} \\ &= \frac{\lambda_1^2-\lambda_1(\alpha_{1,1}+\alpha_{1,2})+\alpha_{1,1}\alpha_{1,2}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} \\ &= \frac{\lambda_1^2-\lambda_1\mathrm{tr}\mathbb{A}_{11}+\det\mathbb{A}_{11}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} && (9.1) \\ |e_{12}|^2 &= \frac{\lambda_1^2-\lambda_1\mathrm{tr}\mathbb{A}_{22}+\det\mathbb{A}_{22}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} && (9.2) \\ |e_{13}|^2 &= \frac{\lambda_1^2-\lambda_1\mathrm{tr}\mathbb{A}_{33}+\det\mathbb{A}_{33}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} && (9.3) \end{aligned}$$

Let us assume, as we may, that $\mathbb{A}$ had been rotationally/unitarily diagonalized:

$$\mathbb{A} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \text{ giving } \begin{cases} \mathbb{A}_{11} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{pmatrix} \\ \mathbb{A}_{22} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{pmatrix} \\ \mathbb{A}_{33} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \end{cases}$$

By (9)

$$\begin{aligned} |e_{11}|^2 = \frac{\lambda_1^2-\lambda_1(\lambda_2+\lambda_3)+\lambda_2\lambda_3}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} &= \frac{\lambda_1^2-(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} \\ &= \frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} = 1 \end{aligned}$$

$$|e_{12}|^2 = \frac{\lambda_1^2 - \lambda_1(\lambda_1+\lambda_3) + \lambda_1\lambda_3}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} = 0$$
$$|e_{13}|^2 = \frac{\lambda_1^2 - \lambda_1(\lambda_1+\lambda_2) + \lambda_1\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} = 0$$

so we have

$$\boldsymbol{e}_1 = \begin{pmatrix} \pm1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{similarly} \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ \pm1 \\ 0 \end{pmatrix}, \; \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ \pm1 \end{pmatrix}$$

The signs can be dismissed as irrelevant over-all signs, so we have (surprisingly?) recovered descriptions of the eigenvectors themselves. The argument extends straightforwardly to arbitrary dimension n, and shows proof of orthogonality to be a simple exercise in the theory of symmetric polynomials. It has led to a result that follows obviously from the assumed diagonal structure of $\mathbb{A}$.

I digress to describe a curious fact. From orthonormal vectors

$$\boldsymbol{e}_i = \begin{pmatrix} e_{i1} \\ e_{i2} \\ e_{i3} \end{pmatrix} \quad : \quad i = 1,2,3$$

construct

$$\mathbb{R} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) = \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{33} \\ e_{13} & e_{23} & e_{33} \end{pmatrix}$$

which is a rotation matrix

$$\mathbb{R}^{\mathsf{T}}\mathbb{R} = \begin{pmatrix} \boldsymbol{e}_1\cdot\boldsymbol{e}_1 & \boldsymbol{e}_1\cdot\boldsymbol{e}_2 & \boldsymbol{e}_1\cdot\boldsymbol{e}_3 \\ \boldsymbol{e}_2\cdot\boldsymbol{e}_1 & \boldsymbol{e}_2\cdot\boldsymbol{e}_2 & \boldsymbol{e}_2\cdot\boldsymbol{e}_3 \\ \boldsymbol{e}_3\cdot\boldsymbol{e}_1 & \boldsymbol{e}_3\cdot\boldsymbol{e}_2 & \boldsymbol{e}_3\cdot\boldsymbol{e}_3 \end{pmatrix} = \mathbb{I}$$

and so gives back the orthonormality statement

$$\sum_{k=1}^{3} e_{ik}e_{jk} = \delta_{ij} \tag{10.1}$$

But so also is $\mathbb{R}^{\mathsf{T}}$ a rotation matrix:

$$\mathbb{R}\,\mathbb{R}^{\mathsf{T}} = \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \end{pmatrix} \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} = \mathbb{I}$$

from which follow the less familiar statements

$$\sum_{i=1}^{3} e_{ij}e_{ik} = \delta_{jk} \tag{10.2}$$

In the cases $j = k$ this reads

$$\sum_{i=1}^{3} e_{ij}e_{ij} = 1 \;\;:\;\; \text{compare the normlization conditions} \;\; \sum_{j=1}^{3} e_{ij}e_{ij} = 1$$

where we are summing corresponding components of different vectors, rather than the various components of a given vector; we have here an analog of what in some contexts is known as a "completeness relation." Returning in this light to applications of the Spectral Construction Theorem, the normalization of $\boldsymbol{e}_1$ asserts

$$\frac{(\lambda_1-\alpha_{1,1})(\lambda_1-\alpha_{1,2})+(\lambda_1-\alpha_{2,1})(\lambda_1-\alpha_{2,2})+(\lambda_1-\alpha_{3,1})(\lambda_1-\alpha_{3,2})}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_2)}=1$$

while the completeness of $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ entails

$$\frac{(\lambda_1-\alpha_{j,1})(\lambda_1-\alpha_{j,2})}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{(\lambda_2-\alpha_{j,1})(\lambda_2-\alpha_{j,2})}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}+\frac{(\lambda_3-\alpha_{j,1})(\lambda_3-\alpha_{j,2})}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}=1$$

In the former we see a shared denominator but all possible α-values in the numerator (it was to manage those that we resorted to the "diagonalization trick"), while in the latter we see only a single pair of α-values in the numerator but an assortment of denominators. The remarkable fact to which I draw attention is that the completeness equation—like also its higher-dimensional siblings—is, according to *Mathematica*, valid as an *identity*—irrespective of the values assigned to the α's.

It is of interest to recall finally that the $\alpha_{a,s}$ that appear in (and bedevil) the Spectral Construction Theorem (8)—eigenvalues of principal submatrices of $\mathbb{A}$—are, as Arthur Cayley (1821–1895) observed, subject to an elegant constraint: the **Cayley interlace theorem**[8] asserts that if $\{\lambda_1,\lambda_2\ldots,\lambda_n\}$ are eigenvalues of an n-dimensional real symmetric/hermitian matrix $\mathbb{A}$, and if $\{\mu_1,\mu_2\ldots,\mu_{n-1}\}$ are eigenvalues of any principal submatrix $\mathbb{M}$ of $\mathbb{A}$, then

$$\lambda_1\leqslant\mu_1\leqslant\lambda_2\leqslant\mu_2\leqslant\lambda_3\leqslant\cdots\leqslant\lambda_{n-1}\leqslant\mu_{n-1}\leqslant\lambda_n$$

The "diagonalization trick" exposed extreme instances of this situation.

Physical origin/application of the Spectral Construction Theorem. I quote from the DPZ paper:[2]

> ABSTRACT: We present a new method of exactly calculating neutrino oscillation probabilities in matter. We show that, given the eigenvalues, all mixing angles follow surprisingly simply and the CP violating phase can also be trivially determined. Then, to avoid the cumbersome expressions for the exact eigenvalues, we have applied previously determined perturbatively approximate eigenvalues to this scheme, and found it to be incredibly precise. We also find that these eigenvalues converge at a rate of five orders of magnitude which is the square of the naive expectation

The "cumbersome expressions" that they—like named predecessors—are at pains to avoid arise from the pedestrian fact that to obtain the eigenvalues

[8] See, for example, Steve Fisk, "A very short proof of Cauchy's interlace theorem for eigenvalues of hermitian matrices," arXiv:math/0502408v1 [math.CA] 18 Feb 2005.

of a 3-dimensional matrix one must solve a cubic polynomial. An equation of the form (8) first appears as equation (2.3) on page 3 of their 17-page paper. The eigenvalues derive exactly/approximately from a 3-dimensional Hamiltonian matrix that purports to describe neutrino oscillation in matter, and the $|\bullet|^2$ on the left describes "the squares of the elements of the mixing matrix." The authors state that "(2.3) is one of the primary results of our paper. Given the eigenvalues of the Hamiltonian and the eigenvalues of the submatrix Hamiltonian, it is possible to write down all nine elements of the mixing matrix, squared. The result is also quite simple and easy to memorize which is contrasted with the complicaterd forms from previous solutions."

The derivation of their version of (8) they reserve for the second of their five appendices. Their argument is specific to the 3-dimensional case, and their notation obscured by their physical preoccupations. Yet DPZ sensed the latent general significane of their result. According to the Quanta Magazine article mentioned on page 1, they "took a chance and contacted Terrance Tao, despite a note on his website warning against such entreaties."[9] Tao responded within two hours, saying he had never seen this before, and included three independent proofs. DPTZ[4]—clearly the work of T—was posted a week later.

ADDENDUM

This record of my effort to understand DPTZ's accomplishment, which was preceded by fairly extensive *Mathematica*-based numerical experimentation, was completed on 5 December. On the 6th I shared it with a few friends who I thought might have interest in it. In the accompanying note I remarked how surprising it is that fresh gems remained to be discovered in a field so old and well plowed as linear algebra, and that the result might well have been discovered 150 years ago by Arthur Cayley, who—on evidence of (for example) his Interlacing Theorem and the Cayley-Hamilton theorem—had a creative interest in (among so many other things) the secret life of eigenvalues.

Today—Pearl Harbor Day—I looked again to the Quanta Magazine article, seeking contact information so that I might commend Natalie Wolchover for her having drawn attention to this obscure work. There I was informed that her November 19th article (now amended) has come to rank as one of the magazine's most popular articles (so much for my delusion that I might be the only person who found it interesting!), and was reminded that soon after the DPTZ arXiv paper[4] was posted Tao had been informed by Jiyuan Zhang that a similar result (in Tao's opinion "similar but not identical") appears already in a paper that he and his senior co-author had posted in June,[10] and that a related formula

[9] Tao may have a reputation for being unapproachable, but his website https://www.math.ucla.edu/~tao suggests quite the opposite.

[10] Peter J. Forrester & J. Zhang, "Co-rank 1 projections and the randomized Horn problem," arXiv:1905.05314v3 [math-ph] 15 June 2019. Those authors are mathematicians at the University of Melbourne, and their work relates to a random matrix problem that Tao and a collaborator had solved in 1999.

appears in the 2001 paper by Yully Baryshnikov which they had taken as their point of departure.

The flood of correspondence provoked by the Quanta article (up-dated on December 4th) led DPTZ to rewrite their original paper. The new paper[11] runs to 26 pages, and provides 50 references. The authors provide about seven alternative proofs of what they now call the **eigenvector-eigenvalue identity**, discussion of the remarkably diverse contexts that have led to its repeated reinvention/application, a graph indicating how references to the identity have been interlinked (sparce because users of the identity have been so often ignorant of each other's work), and in their final §5. SOCIOLOGY OF SCIENCE ISSUES suggest three reasons that the identity has remained so little known:

- *The identity is mostly used as an auxiliary tool for other purposes.*
- *The identity does not have a common name, form or notation, and does not involve uncommon keywords.*
- *The field of linear algebra is too mature, and its domain of applicability is too broad; users rely on textbooks, the content of which is static.*

What they have specifically in mind by those heads is developed in their text.

[11] "Eigenvectors from eigenvalues: survey of a basic identity in linear algebra," arXiv:1908.03795v2 [math-RA] 2 Dec 2019.